Innovative Washing Machine

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Let our polygon be $A_1A_2\dots A_n$.

Let us define the function $f(\varphi)$ as the value of pressure imbalance for the polygon rotated by angle φ.

As the answer to the problem we want to calculate $\frac{1}{2\pi}\int\limits_0^{2\pi} f(\varphi)d\varphi$.

Let us define the function $p_i(\varphi) = -x_i \cdot \sin(\varphi) + y_i \cdot \cos(\varphi)$. It is a coordinate of projection of vertex i into the orthogonal direction to the water level.

If vertices $u_1, u_2, \dots, u_k$ are underwater, when

$$f(\varphi) = \frac{1}{k}\sum_{i=1}^{k}\left(\max_{j=1}^{k} d_{u_j} - d_{u_i}\right) = \frac{1}{k}\sum_{i=1}^{k}\left(p_{u_i}(\varphi) - \min_{j=1}^{k} p_{u_j}(\varphi)\right)$$

$$f(\varphi) = -\left(\frac{1}{k}\sum_{i=1}^{k} x_{u_i}\right)\sin(\varphi) + \left(\frac{1}{k}\sum_{i=1}^{k} y_{u_i}\right)\cos(\varphi) - \min_{j=1}^{k} p_{u_j}(\varphi)$$

Let's note, that $\min\limits_{j=1}^{k} p_{u_j}(\varphi) = \min\limits_{i=1}^{n} p_i(\varphi)$, because the global lowest vertex will be underwater. So:

$$f(\varphi) = -\left(\frac{1}{k}\sum_{i=1}^{k} x_{u_i}\right)\sin(\varphi) + \left(\frac{1}{k}\sum_{i=1}^{k} y_{u_i}\right)\cos(\varphi) - \min_{i=1}^{n} p_i(\varphi)$$

Let us find the integral of $\min\limits_{i=1}^{n} p_i(\varphi)$ and $-\left(\frac{1}{k}\sum\limits_{i=1}^{k} x_{u_i}\right)\sin(\varphi) + \left(\frac{1}{k}\sum\limits_{i=1}^{k} y_{u_i}\right)\cos(\varphi)$ separately.

First part.

We want to calculate $\int\limits_0^{2\pi}\min\limits_{i=1}^{n} p_i(\varphi)d\varphi$. Note, that the point A_i will be lowest for all $\varphi \in [\alpha(\overrightarrow{A_{i-1}A_i}), \alpha(\overrightarrow{A_iA_{i+1}})]$, where $\alpha(\overrightarrow{v})$ is a polar angle of vector $\overrightarrow{v}$.

So the integral is $\sum\limits_{i=1}^{n}\int\limits_{\alpha(\overrightarrow{A_{i-1}A_i})}^{\alpha(\overrightarrow{A_iA_{i+1}})} p_i(\varphi)d\varphi = \sum\limits_{i=1}^{n} y_i \int\limits_{\alpha(\overrightarrow{A_{i-1}A_i})}^{\alpha(\overrightarrow{A_iA_{i+1}})}\cos(\varphi)d\varphi - \sum\limits_{i=1}^{n} x_i \int\limits_{\alpha(\overrightarrow{A_{i-1}A_i})}^{\alpha(\overrightarrow{A_iA_{i+1}})}\sin(\varphi)d\varphi$.

Integration of sin and cos on segment is simple, this sum can be calculated.

Magically, expression can be simplified further, and the integral equals to minus perimeter of the polygon: $-\sum\limits_{i=1}^{n}|A_iA_{i+1}|$.

Second part.

We want to calculate $\int\limits_0^{2\pi}\left(-\left(\frac{1}{k}\sum\limits_{i=1}^{k} x_{u_i}\right)\sin(\varphi) + \left(\frac{1}{k}\sum\limits_{i=1}^{k} y_{u_i}\right)\cos(\varphi)\right)d\varphi$.

During the rotation of the polygon the set of underwater vertices $u_1, u_2, \dots, u_k$ is changing. In which angles the set is changing? Note, that if the set of underwater vertices is changing in some angle φ, one of the vertices A_i lies on the water level in this moment. Vertex A_i can be left or right vertex of the water level segment.

For all vertices A_i let us find the angles $\varphi_{L,i}$, $\varphi_{R,i}$ in which A_i is left and right vertex of the water level segment, respectively. These angles can be found with two pointers method: let us iterate i in clockwise or counterclockwise order and maintain the set of currently underwater vertices (these vertices will be consecutive). So we maintain the pointer to vertex j, such that $area(A_i A_{i+1} \dots A_j) \leq s$ and $area(A_i A_{i+1} \dots A_{j+1}) > s$ (for counterclockwise traversal). This index j can be simply maintained, because we can maintain the sum of pseudoscalar products $\overrightarrow{A_p} \times \overrightarrow{A_{p+1}}$ for $p \in [i, j)$ to maintain the area. After that, to find the angle $\varphi_{L,i}$ we should look at the triangle $\triangle A_i A_j A_{j+1}$ and find a segment that divides the triangle into two pieces with given ratio of areas.

So, such event angles $\varphi_{L,i}$, $\varphi_{R,i}$ where a set of underwater vertices is changing can be found in $O(n)$. After that, we can sort events and calculate our integral, because on each segment between consecutive events, values $\frac{1}{k}\sum_{i=1}^{k} x_{u_i}$, $\frac{1}{k}\sum_{i=1}^{k} y_{u_i}$ are constant and we should just integrate sin and cos.

The total complexity of the solution is $O(n \log n)$.