Cactus Transformation

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Let's initially consider some edge cases:

- If both cactuses are the same, then we don't need to do anything;
- If cactuses consist of only cycles of length 3 and don't contain any bridges, then there is no transformation from the first cactus into the second one because if we remove any edge from the first cactus, we can not add the new one (we need to add the same deleted edge).

For all the other cases, the solution exists. To solve the problem, we will transform both cactuses into the same convenient cactus, let's call it as *star* cactus. If the cactus contains n vertices and c cycles, then the star cactus will have the following cycles: $(1, 2, 3)$, $(1, 4, 5)$, $\ldots$, $(1, 2x, 2x+1)$, all the cycles contain the vertex 1. The bridges of the star cactus will be $(1, 2x+2)$, $(1, 2x+3)$, $\ldots$, $(1, n)$, all the bridges will be incident to the vertex 1.

Operations for transforming the first cactus into the second one will be the concatenation of operations for transforming the first cactus into the star cactus and operations for transforming the second cactus into the star cactus but already in **reverse order**.

So, let's understand how we can transform any cactus into a star cactus. Before understanding the actions needed for an algorithm, let's do some denotations:

- Let's denote by deg_v — the degree of the vertex v $(1 \le v \le n)$;
- Let's denote by $dist_v$ — the minimum distance from the vertex 1 to the vertex v $(1 \le v \le n)$.

First action: let's take any cycle c_1, c_2, ..., c_k of length k (where $k > 3$), and do the following operation on it:

- Remove an edge (c_1, c_k) from the cactus and add an edge (c_1, c_3).

After applying this operation, the number of cycles of length greater than 3 will be decreased by 1. We will perform the following action while the cactus contains any cycle of length greater than 3.

Second action: let's take any bridge (v, u) from the cactus, such that $dist_v < dist_u$, $v \neq 1$, and $u \neq 1$. Remove an edge (v, u) from the cactus and add an edge $(1, u)$. We will perform this action while the cactus contains any bridge that isn't incident to the vertex 1.

Third action: let's take any edge $(1, v)$ that is incident to the vertex 1 and $deg_v > 2$. There are two cases:

Case 1: Edge $(1, v)$ is bridge.

We can note that deg_v is odd because the number of bridges incident to any vertex is at most 1. Since $deg_v > 2$, we know that there is at least one cycle that contains the vertex v. Let's take any cycle that contains the vertex v and assume that the vertices of the cycle are v, u, and w. Let's do the following operations:

- Remove an edge (u, w) and add an edge $(1, u)$;
- Remove an edge (v, w) and add an edge $(1, w)$.

Let's note that after these operations, the number of cycles that contain the vertex 1 will be increased by 1, while the number of bridges incident to the vertex 1 will not be changed.

Case 2: Edge $(1, v)$ isn't bridge.

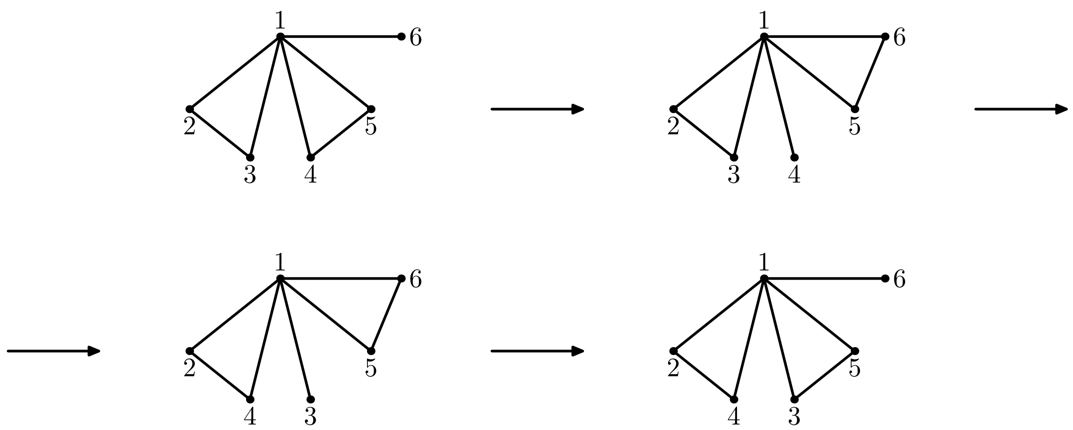
We can note that deg_v is even because all the bridges are incident to the vertex 1. We know that an edge (1, v) isn't a bridge, so let's assume the vertices of the cycle containing an edge (1, v) are 1, v and a. Since $deg_v > 2$, we know that there are at least two cycles that contain the vertex v. So, let's take any cycle that contains the vertex v and is different from the cycle (1, v, a). Let's assume that the vertices of the cycle are v, u, and w. Let's also understand that there is at least one bridge in the cactus because all the cases that don't contain a bridge were considered initially as edge cases. So, let's take some bridge (1, b) incident to the vertex 1. Let's do the following operations:

- Remove an edge (v, a) and add an edge (a, b);
- Remove an edge (u, w) and add an edge (1, u);
- Remove an edge (v, w) and add an edge (1, w).

Let's note that after these operations, the number of cycles that contain the vertex 1 will be increased by 1, while the number of bridges incident to the vertex 1 will not be changed.

We will perform this action with two cases while the cactus contains any vertex $v \neq 1$, such that $deg_v > 2$.

After all of these three actions, we will get the structure of the star cactus, and it remains to fix the labels in the vertices. In the picture below, you can see the operations that are needed for swapping two labels (in the example below, we swapped the labels 3 and 4).



In practice, the number of operations for this solution is approximately 5000, but for solid proof, you can see that it can't be greater than 15000.